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Lie supergroups supported over GL_2 and U_2 associated to the adjoint representation

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Abstract

The algebraic classification of Lie superalgebras based on $\mathfrak{gl}_2$ whose odd module is $\mathfrak{gl}_2$ itself under the adjoint representation, together with the existence and uniqueness theorem for ODE's in supermanifolds, are hereby used as in Lie's theory to produce nonisomorphic Lie supergroups, all of which are supported over the same underlying Lie group GL_2, and whose Lie superalgebras of left-invariant supervector fields are isomorphic to those given by the algebraic classification. Their compact real forms are also studied so as to produce nonisomorphic Lie supergroups supported over the unitary group U_2, and their corresponding maximal tori are described.

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1. Introduction

The main purpose of this work is to describe the Lie supergroup structure of those real and complex Lie supergroups having GL_2 as their underlying Lie group and having a Lie

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superalgebra of the form $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0 = \mathfrak{gl}_2 = \mathfrak{g}_1$, where the action of the even $\mathfrak{gl}_2$ into the odd $\mathfrak{gl}_2$ is given by the adjoint representation (for the basic definitions and main results on Lie superalgebras we follow the standard references: [3,4,9]). It has been proved in [8] that there are 10 (resp., 8) different isomorphism classes of real (resp., complex) Lie superalgebras satisfying these constraints, and it is through each one of them that the different Lie supergroup structures over GL_2 are obtained. In particular, 10 different Lie supergroups based over U_2 are obtained under analogous constraints for their Lie superalgebras.

One physical motivation for understanding precisely these Lie supergroup structures comes from the fact that spacetime is conveniently identified – locally, at least – with the Lie algebra of the unitary group U_2. Therefore, U_2 has naturally associated to it, 10 different possibilities for building up (4, 4)-dimensional models for superspacetimes. Another example of theoretical interest is the Lie superalgebra described by Witten in [12], which can be embedded into one of the nontrivial (4, 4)-dimensional superspacetime models. The associated supergroups will therefore act on their corresponding Lie superalgebras by supersymmetry transformations; thus, under concrete realizations of these Lie supergroups, new light is shed into such transformations.

Now, the problem of obtaining explicitly a Lie supergroup structure over a given Lie group G_0 from its Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, is subtle. One way of doing it is through an algebraic approach using Hopf algebras and smashed products as in [4]. Results obtained this way, however, are not particularly illuminating nor appealing in specific examples, and physical applications call for a somewhat easier to handle machinery and closer to what one is used to from Lie's theory; namely, giving first a faithful representation of the Lie algebra into the Lie algebra of vector fields on some manifold, and obtaining the local coordinate version of the group multiplication law through composition of their integral flows depending on the integration parameters. As far as we know, no detailed and self-consistent account has been given from first principles on how to obtain a Lie supergroup structure in this way.

Now, the main tool in obtaining a Lie group G_0 out of its Lie algebra $\mathfrak{g}_0$ in C^∞ geometry is the existence and uniqueness theorem for ODE's. The problem of putting Lie's theory to work in supermanifolds is that the important results on integral flows of supervector fields and possible supergroup actions defined by them are not just straightforward extensions of their counterparts in smooth manifolds. In fact, it is taken for granted that the integral flow of a smooth vector field on a smooth manifold defines a local action of $\mathbb{R}$ whereas a similar assertion for supermanifolds is much more subtle and elaborate (see [6]).

Besides, we have chosen to follow the approach we have just described, not only to show that Lie's theory works via integral flows etc., but because the realizations of the supergroups and their actions bring us back to a conceptually simpler way of thinking of the supergroups based on GL_2; namely, as 'spaces whose elements are' 2×2 matrices satisfying some invertibility condition and having a 'multiplication law' that gives them a specific group structure. In more detail, we aim to describe the multiplication law in terms of pairs of matrices $(\mathbf{g}, \gamma)$ having $\mathbf{g} \in \mathrm{GL}_2$ (this is the invertibility condition), and γ a 2×2 matrix with entries in the space $\mathcal{S}(E)$ of sections of the GL_2-homogeneous vector bundle $E \to \mathrm{GL}_2$ with typical fiber $\mathfrak{gl}_2$, associated to the adjoint representation $\mathrm{Ad} : \mathrm{GL}_2 \to \mathrm{GL}(\mathfrak{gl}_2)$; in other words, we may write $\gamma \in \mathfrak{gl}_2 \otimes \mathcal{S}(E)$. Now, our findings show that the multiplication law of the various supergroups we study here, can be given in terms of these pairs via an expression of the form:

$$(\mathbf{g}', \gamma') \cdot (\mathbf{g}, \gamma) = (\mathbf{g}'\mathbf{g}(1 + \gamma_0), \gamma + \mathrm{Ad}(\mathbf{g}^{-1})(\gamma' + \gamma_1)), \tag{1.1}$$

where $\gamma_0 \in \mathfrak{gl}_2 \otimes \mathcal{S}(\wedge^0 E \oplus \wedge^2 E \oplus \wedge^4 E)$, $\gamma_1 \in \mathfrak{gl}_2 \otimes \mathcal{S}(\wedge^1 E \oplus \wedge^3 E)$, and $\wedge E = \oplus \wedge^k E$ is the exterior algebra bundle of E; the adjoint action is extended from E to $\wedge E$ in its algebraically natural manner. It is precisely the functional dependence of γ_0, and γ_1 from the entries $(\mathbf{g}', \gamma')$ and $(\mathbf{g}, \gamma)$ what gives the group structure, and what we aim to determine for the various Lie supergroups considered in this work (see Eqs. (3.6) and (3.8), together with Theorem 4.1 for supergroups supported over GL_2; see also Eqs. (5.1) and (5.2) for supergroups supported over U_2, and (6.1) for their maximal tori).

To give a more precise idea of what is involved in the opening paragraph of this introduction, let us first consider the Lie algebra $\mathfrak{gl}_2$ generated, as usual, by the 2×2-matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which we will denote here by x_0, x_1, x_2 and x_3, respectively. In considering the Lie superalgebra $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ we think of the *odd generators* as $P(x_i)$ $(i = 0, 1, 2, 3)$; that is, as elements in the second direct summand $\mathfrak{gl}_2$, which are images under P of the *even* (first direct summand) *generators* x_i, and $P(X)$ stands for X itself with its $\mathbb{Z}_2$-parity reversed. We then extend the map P linearly in such a way that $P \circ P = \mathrm{Id}$ (see [8]). The fact that the action of the even $\mathfrak{gl}_2$ in the odd $\mathfrak{gl}_2$ is given by the adjoint representation is written in terms of P (with a slight abuse of notation) as $P([x_i, x_j]) = [x_i, P(x_j)] = -[P(x_j), x_i]$, where $[\cdot, \cdot]$ stands for the Lie algebra bracket. To complete the Lie superalgebra description, we must give a symmetric bilinear map $\Phi : \mathfrak{gl}_2 \times \mathfrak{gl}_2 \to \mathfrak{gl}_2$ representing the bracket of any pair of odd elements. Writing $\Phi(x_i, x_j)$ for the Lie superalgebra bracket of the two odd elements $P(x_i)$, and $P(x_j)$, it is a straightforward matter to check that the Jacobi identities for the Lie superalgebra imply that (see [8]):

$$\begin{array}{llll} \Phi(x_0, x_0) = \lambda x_0, & \Phi(x_0, x_1) = \mu x_1, & \Phi(x_1, x_1) = 2\nu x_0, & \\ \Phi(x_0, x_2) = \mu x_2, & \Phi(x_1, x_2) = 0, & \Phi(x_2, x_2) = 0, & \\ \Phi(x_0, x_3) = \mu x_3, & \Phi(x_1, x_3) = 0, & \Phi(x_2, x_3) = \nu x_0, & \Phi(x_3, x_3) = 0 \end{array}$$

for arbitrary parameters λ, μ and ν in the ground field. A different symmetric bilinear map $\Phi' : \mathfrak{gl}_2 \times \mathfrak{gl}_2 \to \mathfrak{gl}_2$ would yield a different set of parameters; say λ', μ' and ν', respectively. Let us denote by $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ the $\mathbb{F}$-Lie superalgebra ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ defined by the parameter values (λ, μ, ν), and we have the following statement (see [8]).

Theorem 1.1. *The Lie superalgebras* $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ *and* $\mathfrak{gl}_2(\mathbb{F}; \lambda', \mu', \nu')$ *are isomorphic if and only if there is a Lie algebra automorphism* $T : \mathfrak{gl}_2 \to \mathfrak{gl}_2$ *and an* $\mathbb{F}$-linear isomorphism $S : \mathfrak{gl}_2 \to \mathfrak{gl}_2$ satisfying

$$[T(x), S(y)] = S([x, y]) \quad \textit{and} \quad \Phi'(S(x), S(y)) = T(\Phi(x, y)),$$

for any x and y in the Lie algebra $\mathfrak{gl}_2$. *This is the case if and only if there are nonzero constants a, b and c in the ground field* $\mathbb{F}$, *such that,*

$$\lambda' = \lambda \frac{1}{ab^2}, \qquad \mu' = \mu \frac{1}{abc}, \qquad \nu' = \nu \frac{a}{c^2}.$$

It follows that either, the three parameters λ, μ and ν are equal to zero, or exactly two of them are zero, or exactly one is zero, or none of them is zero. That is how the eight isomorphism classes over $\mathbb{C}$ arise. In the real case, one further sees that the product $\lambda'\nu'$ is equal to $\lambda\nu$ times a positive constant. Therefore, the sign of this product must remain constant, thus giving 10 real isomorphism classes. Concrete representatives for the different classes can be given. We shall agree on choosing the parameter representatives in such a way that, when different from zero, $\lambda = 2$, $\mu = 2$ and $\nu = \pm 1$.

Let us now denote by $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ the Lie supergroup whose underlying Lie group is GL_2 and whose Lie superalgebra is $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$. The supergroup composition law will be obtained from first principles using the ODE theory in supermanifolds developed in [6] and following Lie's original techniques as described above.

Thus, we solve first the problem of giving a faithful representation of the Lie superalgebra $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ inside the Lie superalgebra of supervector fields on some appropriate supermanifold. This is done in Section 2. Then we find in Section 3 the integral flows of the image generators of the representation. The integral flows describe a local action of $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ on the supermanifold that *naturally recovers* the ordinary (and therefore, globally defined) linear action of GL_2 on $\mathbb{F}^2$.

It is perhaps worth investing a little effort in elucidating the precise meaning of the last assertion. We shall follow the basic references on the subject, and understand that a smooth supermanifold structure supported over a given smooth manifold M is sheaf $\mathcal{A}_M$ of associative superalgebras satisfying some specific properties – and one usually refers to the supermanifold $(M, \mathcal{A}_M)$ (see either [4,5], or [11] for details). We would like to focus the reader's attention into one of such defining properties of the structure sheaf $\mathcal{A}_M$; namely, that there is a natural sheaf epimorphism $\mathcal{A}_M \to \mathcal{C}^\infty_M$ onto the sheaf of smooth functions of M. This epimorphism provides a natural forgetful functor from the category of supermanifolds into the category of smooth manifolds that recovers, for each object, the underlying smooth manifold over which the 'super' structure sheaf $\mathcal{A}_M$ is defined. In particular, the natural supermanifolds to consider for faithful representations of $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ into Lie superalgebras of vector fields, are those having $\mathbb{F}^2$ as their underlying manifold. The reason is that if all our constructions are going to be natural, they would have to behave and transform properly under the natural sheaf map $\mathcal{A}_M \to \mathcal{C}^\infty_M$. We would therefore expect the Lie supergroup action of $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ on a supermanifold based over $\mathbb{F}^2$, to project onto the ordinary linear action of GL_2 on $\mathbb{F}^2$. What we have found, however, is that all but two isomorphism classes of the Lie superalgebras we obtained can be represented as supervector fields on the $(2, 2)$-dimensional supermanifold $\mathbb{F}^{2|2}$ in such a way that we can recover the $\mathbb{F}$-linear GL_2-action on $\mathbb{F}^2$. The classes corresponding to $[\lambda \neq 0, \mu = 0, \nu = 0]$ and to $[\lambda = 0, \mu = 0, \nu \neq 0]$ have to be represented as supervector fields in $\mathbb{F}^{3|3}$, and the interpretation of the $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ action is more subtle (see Section 2).

Once we reach the point of identifying the multiplication law $(\mathbf{g}', \gamma') \cdot (\mathbf{g}, \gamma)$ and give a formula for it as in (1.1), we must verify that the Lie superalgebra of left-invariant supervector fields on $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$ is actually isomorphic to the abstract Lie superalgebra $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ we started with. In order to do this we provide an appropriate commutative diagram of supermanifold morphisms capable of stating the left-invariance property. This is a common resource in supermanifold theory: since supermanifold morphisms are not determined by their values on the points of the underlying manifolds involved (see [5]), one must be careful – while remaining strict within the category – each time one needs to leave an argument fixed in a two-argument morphism. An example of the way around this technical issue was given by the 'evaluation map' introduced in [6] in order to deal with the uniqueness of the integral flows of supervector fields. We mention in passing that with a more cautious definition of the category of differentiable supermanifolds, one is able to observe in their morphisms a much more familiar behavior. We refer the reader to [11, p. 138]for a complete discussion of this point, and we take the opportunity to thank the referee for bringing this reference and issue to our attention.

At this point, the classical Lie's theorems find a concrete verification for the Lie supergroups $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$. Other aspects of Lie's theory and some applications can be observed for our particular family of examples by looking at their compact real forms. In order to approach this problem we first consider the real Lie superalgebras $\mathfrak{u}_2(\lambda, \mu, \nu)$ – whose underlying supervector space is $\mathfrak{u}_2 \oplus \mathfrak{u}_2$ – so as to have the even copy of $\mathfrak{u}_2$ generated over the real field by $w_0 = \mathrm{i}I$, $w_3 = \mathrm{i}H$, $w_2 = E - F$ and $w_1 = \mathrm{i}(E + F)$, as usual. We let P be as before, so that the symmetric bilinear equivariant map $\Phi : \mathfrak{u}_2 \times \mathfrak{u}_2 \to \mathfrak{u}_2$ that gives the Lie bracket of any pair of odd elements via $\Phi(z, w) = [P(z), P(w)]$ is

$$\begin{aligned}
&\Phi(w_0, w_0) = \mathrm{i}\lambda w_0, \qquad \Phi(w_0, w_3) = \mathrm{i}\mu w_3, \qquad \Phi(w_3, w_3) = 2\mathrm{i}\nu w_0,\\
&\Phi(w_0, w_2) = \mathrm{i}\mu w_2, \qquad \Phi(w_3, w_2) = 0, \qquad \Phi(w_2, w_2) = 2\mathrm{i}\nu w_0,\\
&\Phi(w_0, w_1) = \mathrm{i}\mu w_1, \qquad \Phi(w_3, w_1) = 0, \qquad \Phi(w_2, w_1) = 0,\\
&\Phi(w_1, w_1) = 2\mathrm{i}\nu w_0.
\end{aligned}$$

Therefore, λ, μ and ν have to be restricted, from taking arbitrary complex values in $\mathfrak{gl}_2(\mathbb{C}; \lambda, \mu, \nu)$, to take only purely imaginary values on $\mathfrak{u}_2(\lambda, \mu, \nu)$. The maximal toral subalgebra of $\mathfrak{u}_2(\lambda, \mu, \nu)$ is generated by w_0, w_3, $P(w_0)$ and $P(w_3)$. We can find the integral flows of the appropriate supervector fields which are images of these generators and also find the composition law for the maximal torus $\mathbb{T}^2(\lambda, \mu, \nu) \subset U_2(\lambda, \mu, \nu)$.

We must mention that we have succeeded in finding out completely general composition laws (i.e., depending on arbitrary values of the parameters (λ, μ, ν)) for four of the eight isomorphism classes of the Lie supergroups $\mathrm{GL}_2(\mathbb{C}; \lambda, \mu, \nu)$. Namely, those having $\nu = 0$. We have been able to give an explicit composition law for a specific representative of the class having $\lambda\mu\nu \neq 0$; namely, for $\mathrm{GL}_2(\mathbb{F}; 2, 2, 1)$ which, as a matter fact, turned out to be a pretty simple one. General composition laws for those $\mathrm{GL}_2(\mathbb{C}; \lambda, \mu, \nu)$ having $\nu \neq 0$, have proved to be difficult to handle. We have, nevertheless succeeded in finding out completely general composition laws for all the maximal tori $\mathbb{T}^2(\lambda, \mu, \nu)$ (i.e., regardless of the parameter values).

The maximal tori arising from the different isomorphism classes of the unitary supergroups brings to the foreground the general problem of classifying all the real Lie supergroup structures that can be defined over the torus $\mathbb{T}^2 = S^1 \times S^1$, whose ‘odd sector’ carries the adjoint representation of $\mathfrak{t}_2 = \mathrm{Lie}(\mathbb{T}^2)$. This problem falls into the general spirit of what has been done in [8], and can be solved by using the same methods; i.e., by classifying first the Lie superalgebra structures on $\mathfrak{t}_2 \oplus \mathfrak{t}_2$ associated to the adjoint representation. Since $\mathfrak{t}_2$ is Abelian, the Jacobi identities for the various combinations of homogeneous elements are all trivial and, therefore, there are no conditions imposed on the symmetric bilinear map $\Phi : \mathfrak{t}_2 \times \mathfrak{t}_2 \to \mathfrak{t}_2$.

Once a basis of $\mathfrak{t}_2$ is given (and the basis of the odd direct summand is the same but with the understanding that its parity has been reversed), the problem of classifying those symmetric bilinear maps $\Phi : \mathfrak{t}_2 \times \mathfrak{t}_2 \to \mathfrak{t}_2$ that yield isomorphic Lie superalgebras on $\mathfrak{t}_2 \oplus \mathfrak{t}_2$ comes down to the problem of classifying pairs (θ^1, θ^2) of real symmetric bilinear forms under the action of $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ given by

$$(T, S) \cdot (\theta^1, \theta^2) = (T_{11} S \cdot \theta^1 + T_{12} S \cdot \theta^2, T_{21} S \cdot \theta^1 + T_{22} S \cdot \theta^2),$$

where T and S belong to $\mathrm{GL}_2(\mathbb{R})$, $S \cdot \theta^i = (S^{-1})^{\mathrm{t}} \theta^i (S^{-1})$ and the indicated matrix entries are referred to the chosen basis. It is proved that there are seven different orbits for this action (only five if one would pose the same problem over the complex field) and it is shown that there is a surjection from the equivalence classes of tori we have found for the various superunitary groups, onto the equivalence classes of Lie superalgebras obtained this way.

2. Representations of $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ by means of supervector fields on supermanifolds

We shall adhere ourselves to the standard references for the basic definitions of Lie superalgebras and their representations (e.g. [2–4,9]), and smooth supermanifolds (e.g. [4,5]). We refer the reader there for the basic definitions and standard conventions.

The aim of this section is to find faithful representations of the Lie superalgebras $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ realized as supervector fields on appropriate supermanifolds.

Let x_0, x_1, x_2, and x_3 be the $\mathfrak{gl}_2$-basis matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and } F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

respectively. Let $y_i = P(x_i)$ $(i = 0, 1, 2, 3)$ and let $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ be generated by the x_i’s in the first direct summand, and the y_i’s in the second. Think of P as built up from the identity map of the underlying vector spaces, so that

$$P = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix},$$

and define the parity $|x_i|$ of the x_i's to be $0 \bmod(2)$, while the parity $|y_i|$ of the y_i's is $1 \bmod(2)$.

Let us assume that $W_0 \oplus W_1$ is a finite-dimensional $\mathbb{Z}_2$-graded vector space, and look at $\mathrm{End}\, W = (\mathrm{End}\, W)_0 \oplus (\mathrm{End}\, W)_1$ with is usual $\mathbb{Z}_2$-grading and Lie superalgebra structure (cf. [9]). Let,

$$x_i \mapsto X_i = \begin{pmatrix} A(x_i) & 0 \\ 0 & D(x_i) \end{pmatrix} \quad \text{and} \quad y_i \mapsto Y_i = \begin{pmatrix} 0 & B(y_i) \\ C(y_i) & 0 \end{pmatrix} \tag{2.1}$$

be a Lie superalgebra homomorphism, so that $A(x_i) \in \mathrm{End}(W_0)$, $D(x_i) \in \mathrm{End}(W_1)$, $B(y_i) \in \mathrm{Hom}(W_1, W_0)$, and $C(y_i) \in \mathrm{Hom}(W_0, W_1)$. In particular, $A : \mathfrak{gl}_2 \to \mathrm{End}(W_0)$, and $D : \mathfrak{gl}_2 \to \mathrm{End}(W_1)$ are ordinary Lie-algebra representations. Now, let $\{e_a\}$ ($a = 1, \ldots, \dim\, W_0$), and $\{f_\mu\}$ ($\mu = 1, \ldots, \dim\, W_1$) be some given bases of W_0, and W_1, respectively. The matrices associated to $A(x_i)$, $B(y_i)$, $C(y_i)$, and $D(x_i)$ allow us to define *even* and *odd* global derivations – X_i, and Y_i, respectively – of the structure sheaf of the $(\dim\, W_0, \dim\, W_1)$-dimensional affine supermanifold $(W_0, C^\infty_{W_0} \otimes \Lambda(W_1^*))$ with (global) coordinates $\{z^a, \zeta^\mu\}$, ($|z^a| = 0$, and $|\zeta^\mu| = 1$), by means of the assignments,

$$x_i \mapsto X_i = \sum_{a,b} A(x_i)_{ab} z^a \frac{\partial}{\partial z^b} + \sum_{\mu,\nu} D(x_i)_{\mu\nu} \zeta^\mu \frac{\partial}{\partial \zeta^\nu},$$

and

$$y_i \mapsto Y_i = \sum_{\mu,b} C(y_i)_{\mu b} \zeta^\mu \frac{\partial}{\partial z^b} + \sum_{a,\nu} B(y_i)_{a\nu} z^a \frac{\partial}{\partial \zeta^\nu}.$$

The global coordinates $\{z^a\}$ and $\{\zeta^\mu\}$ are simply given by the dual bases to $\{e_a\}$ and $\{f_\mu\}$, with the underlying field $\mathbb{F}$ being trivially graded. Straightforward computations show that,

$$\begin{aligned}
[X_i, X_j] &= \sum_{a,b} (A(x_i)A(x_j) - A(x_j)A(x_i))_{ab} z^a \frac{\partial}{\partial z^b} + \sum_{\mu,\nu} (D(x_i)D(x_j) \\
&\quad - D(x_j)D(x_i))_{\mu\nu} \zeta^\mu \frac{\partial}{\partial \zeta^\nu}, \\
[X_i, Y_j] &= \sum_{b,\mu} (D(x_i)C(y_j) - C(y_j)A(x_i))_{\mu b} \zeta^\mu \frac{\partial}{\partial z^b} + \sum_{a,\nu} (A(x_i)B(y_j) \\
&\quad - B(y_j)D(x_i))_{a\nu} z^a \frac{\partial}{\partial \zeta^\nu}, \\
[Y_i, Y_j] &= \sum_{a,b} (B(y_i)C(y_j) + B(y_j)C(y_i))_{ab} z^a \frac{\partial}{\partial z^b} + \sum_{\mu,\nu} (C(y_i)B(y_j) \\
&\quad + C(y_j)B(y_i))_{\mu\nu} \zeta^\mu \frac{\partial}{\partial \zeta^\nu}.
\end{aligned}$$

Since the correspondence (2.1) is a Lie superalgebra homomorphism, it follows that the correspondence $x_i \mapsto X_i$, $y_i \mapsto Y_i$, also defines a Lie superalgebra homomorphism into the Lie superalgebra Der $C^{\infty}_{W_0} \otimes \Lambda(W_1^*)$ of graded derivations of the corresponding affine supermanifold.

Since we want to find faithful representations of the Lie superalgebras $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ realized as supervector fields on supermanifolds over which we can also recover the well-known $\mathrm{GL}_2(\mathbb{F})$-action on a 2-dimensional vector space $\mathbb{F}^2$, it is natural to look first at faithful representations of the form (2.1) in $W_0 \oplus W_1 = \mathbb{F}^2 \oplus \mathbb{F}^2$. In doing this we have found faithful representations only for some isomorphism classes of the Lie superalgebras $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$. For some other classes it has been necessary to take $W_0 \oplus W_1 = \mathbb{F}^3 \oplus \mathbb{F}^3$. Our results are given in the following lemma.

Lemma 2.1.

(1) *Let the ground field be $\mathbb{R}$. Lie superalgebras in the equivalence classes of*

$$\lambda\nu > 0,\ \mu \neq 0, \quad \lambda\nu < 0,\ \mu \neq 0, \quad \lambda\nu < 0,\ \mu = 0, \quad \nu\mu \neq 0,\ \lambda = 0,$$
$$\lambda\mu \neq 0,\ \nu = 0, \quad \mu \neq 0,\ \lambda = \nu = 0, \quad \lambda = \mu = \nu = 0$$

admit a matrix representation in the supervector space $\mathbb{R}^2 \oplus \mathbb{R}^2$ of the following form:

$$X \in \mathfrak{gl}_2,\ |X| = 0, \quad \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \qquad Y \in \mathfrak{sl}_2,\ |Y| = 1, \quad \begin{pmatrix} 0 & dY \\ eY & 0 \end{pmatrix},$$
$$I \in \mathfrak{gl}_2,\ |I| = 1, \quad \begin{pmatrix} 0 & gI \\ kI & 0 \end{pmatrix},$$

where $\lambda = 2gk$, $\mu = eg + dk$ and $\nu = ed$. The real Lie superalgebras lying in the equivalence class of $\lambda\nu > 0$ and $\mu = 0$ admit a similar matrix representation, but in the supervector space $\mathbb{C}^2 \oplus \mathbb{C}^2$.

(2) *Let the ground field be $\mathbb{C}$. Lie superalgebras in the equivalence classes of*

$$\lambda\nu \neq 0,\ \mu \neq 0, \quad \lambda\nu \neq 0,\ \mu = 0, \quad \nu\mu \neq 0,\ \lambda = 0, \quad \lambda\mu \neq 0,\ \nu = 0,$$
$$\mu \neq 0,\ \lambda = \nu = 0, \quad \lambda = \mu = \nu = 0$$

admit a matrix representation of the type above in the supervector space $\mathbb{C}^2 \oplus \mathbb{C}^2$.

In either case, their explicit realizations in terms of supervector fields in the supermanifold $\mathbb{R}^{2|2}$ or *$\mathbb{C}^{2|2}$ with the global coordinates $\{z^1, z^2; \zeta^1, \zeta^2\}$ described above are given by*

$$X_0 = z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} + \zeta^2 \frac{\partial}{\partial \zeta^2},$$
$$X_1 = z^1 \frac{\partial}{\partial z^1} - z^2 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^1} - \zeta^2 \frac{\partial}{\partial \zeta^2}, \qquad X_2 = z^1 \frac{\partial}{\partial z^2} + \zeta^1 \frac{\partial}{\partial \zeta^2},$$
$$X_3 = z^2 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial \zeta^1}, \qquad Y_0 = k\left(\zeta^1 \frac{\partial}{\partial z^1} + \zeta^2 \frac{\partial}{\partial z^2}\right) + g\left(z^1 \frac{\partial}{\partial \zeta^1} + z^2 \frac{\partial}{\partial \zeta^2}\right),$$

$$Y_1 = e\left(\zeta^1\frac{\partial}{\partial z^1} - \zeta^2\frac{\partial}{\partial z^2}\right) + d\left(z^1\frac{\partial}{\partial \zeta^1} - z^2\frac{\partial}{\partial \zeta^2}\right),$$

$$Y_2 = e\zeta^1\frac{\partial}{\partial z^2} + dz^1\frac{\partial}{\partial \zeta^2}, \qquad Y_3 = e\zeta^2\frac{\partial}{\partial z^1} + dz^2\frac{\partial}{\partial \zeta^1}.$$

(3) *Let the ground field* $\mathbb{F}$ *be either* $\mathbb{R}$ *or* $\mathbb{C}$. *Lie superalgebras in the equivalence class of* $\nu \neq 0$ *with* $\mu = \lambda = 0$ *can be faithfully represented in terms of supervector fields in the supermanifold* $\mathbb{F}^{3|3}$ *with coordinates* $\{z^0, z^1, z^2; \zeta^0, \zeta^1, \zeta^2\}$ *given by the same expressions for* X_k $(k = 0, 1, 2, 3)$ *and* Y_ℓ $(\ell = 1, 2, 3)$ *in* (1) *or* (2) *above* (*corresponding to the parameter values* $d = 1$ *and* $e = \nu$), *together with,*

$$Y_0 = z^0\frac{\partial}{\partial \zeta^0}.$$

(4) *Let the ground field* $\mathbb{F}$ *be either* $\mathbb{R}$ *or* $\mathbb{C}$. *Lie superalgebras in the equivalence class of* $\lambda \neq 0$ *with* $\mu = \nu = 0$ *can be faithfully represented in terms of supervector fields in the supermanifold* $\mathbb{F}^{3|3}$ *with coordinates* $\{z^0, z^1, z^2; \zeta^0, \zeta^1, \zeta^2\}$ *given by the same expressions for* X_k $(k = 1, 2, 3)$ *and* Y_ℓ $(\ell = 1, 2, 3)$ *in* (1) *or* (2) *above* (*corresponding to the parameter values* $d = 1$ *and* $e = 0$), *together with,*

$$X_0 = z^0\frac{\partial}{\partial z^0} + \zeta^0\frac{\partial}{\partial \zeta^0}$$

and

$$Y_0 = \zeta^0\frac{\partial}{\partial z^0} + \frac{\lambda}{2}z^0\frac{\partial}{\partial \zeta^0}.$$

Proof. We first consider $V_0 = V_1 = \mathbb{F}^3$ and a 3-dimensional representation

$$A = \rho_{(a,b,c)} : \mathfrak{gl}_2 \to \operatorname{End}\mathbb{F}^3$$

depending on the parameters $(a, b, c) \in \mathbb{F}^3$, where

$$\rho_{(a,b,c)}(x_0) = \begin{pmatrix} a & c(b-a) & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad \rho_{(a,b,c)}(x_1) = \begin{pmatrix} 0 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\rho_{(a,b,c)}(x_2) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho_{(a,b,c)}(x_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly, we consider

$$D = \rho_{(a',b',c')} : \mathfrak{gl}_2 \to \operatorname{End}\mathbb{F}^3.$$

From the conditions that the odd module is equal to the adjoint representation, it is easy to check that

$$B(y_i) = d\rho_{(a,b,c)}(y_i) \quad \text{and} \quad C(y_i) = e\rho_{(a',b',c')}(y_i), \quad i = 1, 2, 3,$$

whereas

$$B(y_0) = \begin{pmatrix} f & cg - c'f & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}, \qquad C(y_0) = \begin{pmatrix} h & c'k - ch & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

Therefore, the equations that have to be satisfied for these matrices to define a representation of $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu)$ are

$$2fh = \lambda a = \lambda a', \qquad 2gk = \lambda b = \lambda b', \qquad \mu = eg + dk$$

and

$$\nu a = \nu a' = 0, \qquad \nu b = \nu b' = ed.$$

We can now proceed to see how the concrete representatives given in Theorem 1.1 can be realized via this family of representations. The equations that have to be solved are posed as follows:

$$\lambda\nu \neq 0,\ \mu \neq 0, \quad \mu = eg + dk, \qquad a = a' = fh = 0, \qquad b = b' = \frac{2gk}{\lambda} = \frac{ed}{\nu};$$

$$\lambda\nu \neq 0,\ \mu = 0, \quad 0 = eg + dk, \qquad a = a' = fh = 0, \qquad b = b' = \frac{2gk}{\lambda} = \frac{ed}{\nu};$$

$$\nu\mu \neq 0,\ \lambda = 0, \quad \mu = eg + dk, \qquad a = a' = fh = 0, \qquad b = b' = \frac{ed}{\nu},$$

$$gk = 0; \qquad \lambda\mu \neq 0,\ \nu = 0, \quad \mu = eg + dk, \qquad a = a' = \frac{2fh}{\lambda},$$

$$b = b' = \frac{2gk}{\lambda}, \qquad ed = 0; \qquad \nu \neq 0,\ \mu = \lambda = 0, \qquad 0 = eg + dk,$$

$$a = a' = fh = 0, \qquad b = b' = \frac{2gk}{\lambda} = \frac{ed}{\nu}; \qquad \lambda \neq 0,\ \mu = \nu = 0,$$

$$0 = eg + dk, \qquad a = a' = \frac{2fh}{\lambda}, \qquad b = b' = \frac{2gk}{\lambda}, \qquad ed = 0;$$

$$\mu \neq 0,\ \lambda = \nu = 0, \quad \mu = eg + dk, \qquad fh = 0, \qquad gk = 0, \qquad ed = 0;$$

$$\lambda = \mu = \nu = 0, \quad 0 = eg + dk, \qquad fh = 0, \qquad gk = 0, \qquad ed = 0.$$

A word must be said about the class of $\lambda\nu > 0$ and $\mu = 0$ over the reals. It is a straightforward matter to see that the equations to be solved require imaginary numbers. That means that the Lie superalgebras coming from that class need to be represented on a complex supermanifold, which nevertheless may be regarded as a real supermanifold with twice as many even and odd dimensions. □

3. The associated Lie supergroups $\mathrm{GL}_2(\mathbb{F}; \lambda, \mu, \nu)$

We now proceed to find the integral flows of each of the represented supervector fields using the theory and techniques introduced in [6]. We shall start with those supervector fields X_i and Y_i from Lemma 2.1 that can be realized in the (2, 2)-dimensional supermanifolds $\mathbb{F}^{2|2}$. Thus, let $\Gamma_{x_i} : \mathbb{R}^{1|1} \times \mathbb{F}^{2|2} \to \mathbb{F}^{2|2}$ be the integral flow of the *even* supervector field X_i that represents x_i. According to [6] (see Proposition 3.2 therein), the algebra morphism $\Gamma^*_{x_i} : C^\infty(\mathbb{F}^2) \otimes \Lambda((\mathbb{F}^2)^*) \to C^\infty(\mathbb{R} \times \mathbb{F}^2) \otimes \Lambda((\mathbb{R} \oplus \mathbb{F}^2)^*)$ that solves the ODE posed by the vector field X_i is explicitly given by $\mathrm{Exp}(tX_i) = \mathrm{Id}_{C^\infty(\mathbb{F}^2)} + tX + \frac{t^2}{2!} X_i \circ X_i + \cdots$, where $t \in \mathbb{R}$ is the *even parameter* resulting from the integration process of the ODE. We shall distinguish among the integration parameters t that result from the ODE's posed by each vector field, by writing $\Gamma^*_{x_i} = \mathrm{Exp}(t_i X_i)$, $t_i \in \mathbb{R}$ $(i = 0, 1, 2, 3)$.

By explicitly computing the effect of $\mathrm{Exp}(t_i X_i)$ on the coordinates $z^1, z^2, \zeta^1, \zeta^2$, we find that,

$$\Gamma^*_{x_0} = \mathrm{Exp}(t_0 X_0) : \begin{cases} z^1 \mapsto \mathrm{e}^{t_0} z^1 \\ z^2 \mapsto \mathrm{e}^{t_0} z^2 \\ \zeta^1 \mapsto \mathrm{e}^{t_0} \zeta^1 \\ \zeta^2 \mapsto \mathrm{e}^{t_0} \zeta^2 \end{cases}, \qquad \Gamma^*_{x_1} = \mathrm{Exp}(t_1 X_1) : \begin{cases} z^1 \mapsto \mathrm{e}^{t_1} z^1 \\ z^2 \mapsto \mathrm{e}^{-t_1} z^2 \\ \zeta^1 \mapsto \mathrm{e}^{t_1} \zeta^1 \\ \zeta^2 \mapsto \mathrm{e}^{-t_1} \zeta^2 \end{cases},$$

$$\Gamma^*_{x_2} = \mathrm{Exp}(t_2 X_2) : \begin{cases} z^1 \mapsto z^1 \\ z^2 \mapsto z^2 + t_2 z^1 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 + t_2 \zeta^1 \end{cases}, \qquad \Gamma^*_{x_3} = \mathrm{Exp}(t_3 X_3) : \begin{cases} z^1 \mapsto z^1 + t_3 z^2 \\ z^2 \mapsto z^2 \\ \zeta^1 \mapsto \zeta^1 + t_3 \zeta^2 \\ \zeta^2 \mapsto \zeta^2 \end{cases}$$

It is easily seen that these transformations correspond precisely with what one would obtain under the identifications,

$$z^1 \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad z^2 \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \zeta^1 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \zeta^2 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

by acting on the indicated 4×1 'unit columns' with each of the 4×4 matrices $\mathrm{Exp}(t_i X_i)$ $(i = 0, 1, 2, 3)$ that are obtained when X_i gets identified with the 4×4 matrix associated to x_i via the representation (2.1) explicitly described in Lemma 2.1. We thus set up the correspondences,

$$\Gamma^*_{x_0} \leftrightarrow \begin{pmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{t_0} & 0 & 0 \\ 0 & 0 & e^{t_0} & 0 \\ 0 & 0 & 0 & e^{t_0} \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{t_0} \end{pmatrix} = \mathfrak{m}(t_0; x_0),$$

$$\Gamma^*_{x_1} \leftrightarrow \begin{pmatrix} e^{t_1} & 0 & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1} & 0 \\ 0 & 0 & 0 & e^{-t_1} \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix} = \mathfrak{m}(t_1; x_1),$$

$$\Gamma_{x_2}{}^* \leftrightarrow \begin{pmatrix} 1 & t_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} = \mathfrak{m}(t_2; x_2),$$

$$\Gamma^*_{x_3} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t_3 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ t_3 & 1 \end{pmatrix} = \mathfrak{m}(t_3; x_3).$$

We can now compose any two morphisms in some prescribed order in order to see what the effect of the composition is, and to identify the final result with *the rule* to compose the group *even coordinates* t_0, t_1, t_2 and t_3. That is,

$$\mathfrak{m}(t_i; x_i) \cdot \mathfrak{m}(t'_j; x_j) \quad \{\text{must correspond to the composition of}\} \quad \Gamma^*_{x_i} = \operatorname{Exp}(t_i X_i)$$
$$\text{and} \quad \Gamma^*_{x_j} = \operatorname{Exp}(t'_j X_j)$$

in some appropriate order. By computing directly with the integral flows $\Gamma^*_{x_i} = \operatorname{Exp}(t_i X_i)$, where the X_i are taken as the even supervector fields in Lemma 2.1, we see that, the appropriate order is

$$\mathfrak{m}(t_i; x_i) \cdot \mathfrak{m}(t'_j; x_j) \leftrightarrow \operatorname{Exp}(t_i X_i) \circ \operatorname{Exp}(t'_j X_j)$$

because it is in this, and only this way, that the composition law for the parameters t_i, expressed in matrix form as above, actually corresponds to the usual rule for matrix multiplication.

More generally, we may perform a change of parameters and transform $\mathrm{t} = (t_0, t_1, t_2, t_3)$ into a new set of parameters $\mathbf{g} = (\alpha, \beta, \gamma, \delta)$ in such way that if

$$\operatorname{Exp}(t_0 X_0) \circ \operatorname{Exp}(t_1 X_1) \circ \operatorname{Exp}(t_2 X_2) \circ \operatorname{Exp}(t_3 X_3) = \Gamma^*_{\mathbf{g}}$$

then,

$$\Gamma_{\mathbf{g}}^{*}:\begin{cases} z^1 \mapsto \alpha z^1+\gamma z^2\\ z^2 \mapsto \beta z^1+\delta z^2\\ \zeta^1 \mapsto \alpha\zeta^1+\gamma\zeta^2\\ \zeta^2 \mapsto \beta\zeta^1+\delta\zeta^2\end{cases}$$

That is,

$$\alpha=(1+t_2t_3)\,\mathrm{e}^{t_0+t_1},\qquad \beta=t_2\,\mathrm{e}^{t_0+t_1},\qquad \gamma=t_3\,\mathrm{e}^{t_0-t_1}\quad\text{and}\quad \delta=\mathrm{e}^{t_0-t_1}.$$

Therefore, from

$$\mathrm{Exp}(t'_0X_0)\circ\mathrm{Exp}(t'_1X_1)\circ\mathrm{Exp}(t'_2X_2)\circ\mathrm{Exp}(t'_3X_3)=\Gamma_{\mathbf{g}'}^{*}$$

and $\Gamma_{\mathbf{g}''}^{*}=\Gamma_{\mathbf{g}'}^{*}\circ\Gamma_{\mathbf{g}}^{*}$, one concludes that

$$\mathbf{g}'\leftrightarrow\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix},\qquad \mathbf{g}\leftrightarrow\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\Rightarrow\mathbf{g}''\leftrightarrow\begin{pmatrix}\alpha'\alpha+\beta'\gamma & \alpha'\beta+\beta'\delta\\ \gamma'\alpha+\delta'\gamma & \gamma'\beta+\delta'\delta\end{pmatrix}.$$

Remark. What we have accomplished by proceeding this way is to actually recover the (local) Lie group multiplication law between any two generators in the identity component. Note that this procedure only yields a *multiplication table* for the group generators. However, this table has been obtained from the actual composition of integral flows, by recording the overall effect on the local coordinates. Therefore, the multiplication law obtained this way is associative. Finally, by going into the group ring associated to this multiplication law, and writing down the general 2×2 matrix in the usual form (in terms of new coordinate parameters), one recovers (locally) the usual law for matrix multiplication as the associated group operation. Now, the question of whether the matrix composition law we obtained is globally defined or not on the whole underlying group GL_2, is purely topological. It only depends on what happens at the level of the Lie (sub)algebra one is to *integrate* or *exponentiate* up to a local group, and the actual group one wants to get at. In particular, what we have already done for the even generators clearly recovers the ordinary multiplication law of GL_2, which we already know is globally defined. The point is that the introduction of the odd generators of the Lie superalgebra does not alter this fact. This has been discussed and elucidated in Theorem 6 and Corollary 9 of [7].

Even though this remark is very well understood in the classical Lie theory, we now want to see how the quoted results from [7] get realized when we include the contributions coming from the integral flows of the odd vector fields representing the odd Lie algebra generators y_0, y_1, y_2 and y_3. As mentioned before, the techniques introduced in [6] can be readily applied and in this case, the integral flow $\Gamma_{y_i}:\mathbb{R}^{1|1}\times\mathbb{F}^{2|2}\to\mathbb{F}^{2|2}$ depends on an odd parameter τ_i, as (cf. Lemma 3.3 in [6]) $\Gamma_{y_i}^{*}=\mathrm{Exp}(\tau_iY_i)=\mathrm{i}d+\tau_iY_i$. We may then immediately compute its effect on the coordinates z^1, z^2, ζ^1, ζ^2, and obtain

$$\Gamma^*_{y_0} = \mathrm{Exp}(\tau_0 Y_0) : \begin{cases} z^1 \mapsto z^1 + k\tau_0\zeta^1 \\ z^2 \mapsto z^2 + k\tau_0\zeta^2 \\ \zeta^1 \mapsto \zeta^1 + g\tau_0 z^1 \\ \zeta^2 \mapsto \zeta^2 + g\tau_0 z^2 \end{cases},$$

$$\Gamma^*_{y_1} = \mathrm{Exp}(\tau_1 Y_1) : \begin{cases} z^1 \mapsto z^1 + e\tau_1\zeta^1 \\ z^2 \mapsto z^2 - e\tau_1\zeta^2 \\ \zeta^1 \mapsto \zeta^1 + d\tau_1 z^1 \\ \zeta^2 \mapsto \zeta^2 - d\tau_1 z^2 \end{cases},$$

$$\Gamma^*_{y_2} = \mathrm{Exp}(\tau_2 Y_2) : \begin{cases} z^1 \mapsto z^1 \\ z^2 \mapsto z^2 + e\tau_2\zeta^1 \\ \zeta^1 \mapsto \zeta^1 \\ \zeta^2 \mapsto \zeta^2 + d\tau_2 z^1 \end{cases},$$

$$\Gamma^*_{y_3} = \mathrm{Exp}(\tau_3 Y_3) : \begin{cases} z^1 \mapsto z^1 + e\tau_3\zeta^2 \\ z^2 \mapsto z^2 \\ \zeta^1 \mapsto \zeta^1 + d\tau_3 z^2 \\ \zeta^2 \mapsto \zeta^2 \end{cases}.$$

In order to find the multiplication law for the supergroup in terms of its own local coordinates (actually, the integration parameters t_i and τ_i), we choose a definite sequence for the integral flows: we shall write

$$\Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3) := \Gamma^*_{\mathbf{g}} \circ \Gamma^*_{y_0} \circ \Gamma^*_{y_1} \circ \Gamma^*_{y_2} \circ \Gamma^*_{y_3}$$

and, from

$$\Psi(\mathbf{g}''; \tau''_0, \tau''_1, \tau''_2, \tau''_3) = \Psi(\mathbf{g}'; \tau'_0, \tau'_1, \tau'_2, \tau'_3) \circ \Psi(\mathbf{g}; \tau_0, \tau_1, \tau_2, \tau_3), \tag{3.1}$$

we shall use this equation to find the Lie supergroup multiplication law and cast it in the form

$$(\mathbf{g}'', \boldsymbol{\tau}'') = (\mathbf{g}', \boldsymbol{\tau}') \cdot (\mathbf{g}, \boldsymbol{\tau}) \tag{3.2}$$

as in (1.1). For the sake of illustration, let us first compute the composition law for the integral flows depending on the odd generators. We get

$$\mathrm{Exp}(\tau_2 Y_2) \circ \mathrm{Exp}(\tau_3 Y_3) : \begin{cases} z^1 \mapsto \mathrm{Exp}(\tau_2 Y_2)(z^1 + e\tau_3\zeta^2) \\ z^2 \mapsto \mathrm{Exp}(\tau_2 Y_2)(z^2) \\ \zeta^1 \mapsto \mathrm{Exp}(\tau_2 Y_2)(\zeta^1 + d\tau_3 z^2) \\ \zeta^2 \mapsto \mathrm{Exp}(\tau_2 Y_2)(\zeta^2) \end{cases}$$

We shall do this carefully only once so that the reader can see what is involved: using the fact that that $\mathrm{Exp}(\tau_2 Y_2) = \mathrm{i}d + \tau_2 Y_2$ and the fact that Y_2 is an odd derivation, we get

$$\begin{aligned}\mathrm{Exp}(\tau_2 Y_2)(z^1 + e\tau_3\zeta^2) &= z^1 + e\tau_3\zeta^2 + \tau_2 Y_2\,(z^1 + e\tau_3\zeta^2)\\ &= z^1 + e\tau_3\zeta^2 + \tau_2 Y_2\,(z^1) + \tau_2 Y_2\,(e\tau_3\zeta^2)\\ &= z^1 + e\tau_3\zeta^2 - e\tau_2 Y_2(\zeta^2\tau_3)\\ &= z^1 + e\tau_3\zeta^2 - e\tau_2(Y_2(\zeta^2)\tau_3 - \zeta^2 Y_2(\tau_3))\\ &= z^1 + e\tau_3\zeta^2 - e\tau_2 dz^1\tau_3 = z^1 + e\tau_3\zeta^2 - ed\tau_2\tau_3 z^1.\end{aligned}$$

Note that we have used the fact that $Y_2(\tau_3) = 0$. The final result shows that $\mathrm{Exp}(\tau_2 Y_2)(e\tau_3\zeta^2) = e\tau_3\,\mathrm{Exp}(\tau_2 Y_2)(\zeta^2)$ as it should be, since for each fixed value of the odd section τ_2, $\mathrm{Exp}(\tau_2 Y_2)$ must be an algebra isomorphism and, therefore, the constants – even the odd constants like τ_3 – must be preserved by it. At the light of this, it is very easy to prove that

$$\mathrm{Exp}(\tau_2 Y_2)\circ\mathrm{Exp}(\tau_3 Y_3) : \begin{cases} z^1 \mapsto (1 - ed\tau_2\tau_3)z^1 + e\tau_3\zeta^2\\ z^2 \mapsto z^2 + e\tau_2\zeta^1\\ \zeta^1 \mapsto (1 - ed\tau_2\tau_3)\zeta^1 + d\tau_3 z^2\\ \zeta^2 \mapsto \zeta^2 + d\tau_2 z^1.\end{cases}$$

It is a straightforward computation to show that in writing Ψ, $\mathbf{z}$ and ζ as a shorthand notation for $\Psi(\mathbf{g};\tau_0,\tau_1,\tau_2,\tau_3)$,

$$\mathbf{z} = \begin{pmatrix} z^1\\ z^2\end{pmatrix} \text{ and } \zeta = \begin{pmatrix}\zeta^1\\ \zeta^2\end{pmatrix},$$

respectively,

$$\Psi\mathbf{z} = \boldsymbol{A}\mathbf{z} + \boldsymbol{C}\zeta \quad\text{and}\quad \Psi\zeta = \boldsymbol{B}\mathbf{z} + \boldsymbol{D}\zeta \tag{3.3}$$

where $\boldsymbol{A}$ and $\boldsymbol{D}$ are invertible matrices with even entries, whereas $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices with odd entries; actually, the explicit values for these matrices are

$$\boldsymbol{A} = \mathbf{g}\begin{pmatrix}(1 - eg\tau_0\tau_1)(1 - ed\tau_2\tau_3) & -e(g\tau_0 + d\tau_1)\tau_2\\ -e(g\tau_0 - d\tau_1)\tau_3 & 1 + eg\tau_0\tau_1\end{pmatrix},$$

$$\boldsymbol{B} = \mathbf{g}\begin{pmatrix}(g\tau_0 + d\tau_1)(1 - ed\tau_2\tau_3) & d(1 - eg\tau_0\tau_1)\tau_2\\ d(1 + eg\tau_0\tau_1)\tau_3 & g\tau_0 - d\tau_1\end{pmatrix},$$

$$\boldsymbol{C} = \mathbf{g}\begin{pmatrix}(k\tau_0 + e\tau_1)(1 - ed\tau_2\tau_3) & e(1 - kd\tau_0\tau_1)\tau_2\\ e(1 + kd\tau_0\tau_1)\tau_3 & k\tau_0 - e\tau_1\end{pmatrix},$$

$$\boldsymbol{D} = \mathbf{g} \begin{pmatrix} (1 - dk\tau_0\tau_1)(1 - ed\tau_2\tau_3) & -d(k\tau_0 + e\tau_1)\tau_2 \\ -d(k\tau_0 - e\tau_1)\tau_3 & 1 + dk\tau_0\tau_1 \end{pmatrix}.$$

As we mentioned before, we shall deduce the multiplication law from (3.1) and, using similar expressions as (3.3) for Ψ' and Ψ'', we check that (3.1) implies

$$\begin{aligned} &\boldsymbol{A}'' = \boldsymbol{A}'\boldsymbol{A} - \boldsymbol{B}'\boldsymbol{C}, \qquad \boldsymbol{B}'' = \boldsymbol{A}'\boldsymbol{B} + \boldsymbol{B}'\boldsymbol{D}, \qquad \boldsymbol{C}'' = \boldsymbol{C}'\boldsymbol{A} + \boldsymbol{D}'\boldsymbol{C}, \\ &\boldsymbol{D}'' = -\boldsymbol{C}'\boldsymbol{B} + \boldsymbol{D}'\boldsymbol{D}. \end{aligned} \tag{3.4}$$

Remark. This matrix product is given in [10] and is the one which corresponds to the composition law for two endomorphism on the graded vector space of dimension (n, n).

Let us consider $\nu = 0$ (e.g., choosing $e = 0$). Then (3.4) implies that

$$\begin{aligned} &\mathbf{g}'' = \mathbf{g}'\mathbf{g} + k\tau_0\mathbf{g}'(g\tau_0'\mathbb{1} + d\boldsymbol{\tau}')\mathbf{g}, \qquad \tau_0'' = \tau_0' + \tau_0, \\ &\boldsymbol{\tau}'' = \boldsymbol{\tau} + \mathbf{g}^{-1}\boldsymbol{\tau}'\mathbf{g} - dk\tau_0(\mathbf{g}^{-1}\boldsymbol{\tau}'\mathbf{g})^2, \end{aligned} \tag{3.5}$$

where $\mathbb{1}$ stands for the 2×2 identity matrix,

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_3 & -\tau_1 \end{pmatrix}$$

and similar expressions for $\boldsymbol{\tau}'$ and $\boldsymbol{\tau}''$. This yields the product in the corresponding supergroup in the schematic form (1.1) where,

$$\begin{aligned} &\gamma_0 = \left(\mu - \frac{\lambda}{2}\right)\left(\frac{\gamma_{11}' + \gamma_{22}'}{2}\right)\left(\frac{\gamma_{11} + \gamma_{22}}{2}\right)\mathbb{1} + \mu\left(\frac{\gamma_{11} + \gamma_{22}}{2}\right)\mathrm{Ad}(\mathbf{g}^{-1})\gamma', \\ &\gamma_1 = -\mu\left(\frac{\gamma_{11} + \gamma_{22}}{2}\right)(\gamma')^2, \end{aligned} \tag{3.6}$$

where

$$\gamma = \begin{pmatrix} \tau_0 + \tau_1 & \tau_2 \\ \tau_3 & \tau_0 - \tau_1 \end{pmatrix}.$$

It is a straightforward matter to check that the associativity law holds true for this multiplication (although this was something we already knew by first principles). Note that $(\mathbb{1}, \mathbf{0})$ is the supergroup's identity element, where $\mathbf{0}$ is the 2×2 zero matrix. A straightforward computation shows that the inverse element for $(\mathbf{g}, \gamma)$, which we shall write as $(\mathbf{g}, \gamma)^{-1}$, is given by

$$(\mathbf{g},\gamma)^{-1}=\left(\mathbf{g}^{-1}+\mu\left(\frac{\gamma_{11}+\gamma_{22}}{2}\right)\gamma\mathbf{g}^{-1},-\mathbf{g}\gamma\mathbf{g}^{-1}+\mu\left(\frac{\gamma_{11}+\gamma_{22}}{2}\right)\left(\mathbf{g}\gamma\mathbf{g}^{-1}\right)^{2}\right).$$

The case with $\nu=0$ given by (4) of Lemma 2.1 can be computed as follows: first, from $\Gamma^*_{\mathbf{g}''}=\Gamma^*_{\mathbf{g}'}\circ\Gamma^*_{\mathbf{g}}$ one concludes that

$$\mathbf{g}'\leftrightarrow\begin{pmatrix}\epsilon' & 0 & 0\\ 0 & \alpha' & \beta'\\ 0 & \gamma' & \delta'\end{pmatrix},$$

$$\mathbf{g}\leftrightarrow\begin{pmatrix}\epsilon & 0 & 0\\ 0 & \alpha & \beta\\ 0 & \gamma & \delta\end{pmatrix}\Rightarrow\mathbf{g}''\leftrightarrow\begin{pmatrix}\epsilon'\epsilon & 0 & \\ 0 & \alpha'\alpha+\beta'\gamma & \alpha'\beta+\beta'\delta\\ 0 & \gamma'\alpha+\delta'\gamma & \gamma'\beta+\delta'\delta\end{pmatrix},$$

where

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{SL}_2,$$

and $\epsilon=\det\mathbf{g}$. We then write $\Psi(\mathbf{g};\tau_0,\tau_1,\tau_2,\tau_3):=\Gamma^*_{\mathbf{g}}\circ\Gamma^*_{y_0}\circ\Gamma^*_{y_1}\circ\Gamma^*_{y_2}\circ\Gamma^*_{y_3}$ as before, where $\Gamma^*_{y_i}$ is the integral flow of the odd vector field representing the odd Lie algebra generator y_i. If

$$\mathbf{z}=\begin{pmatrix}z^0\\ z^1\\ z^2\end{pmatrix}\ \text{and}\ \zeta=\begin{pmatrix}\zeta^0\\ \zeta^1\\ \zeta^2\end{pmatrix},$$

then the analogue of (3.3) implies that

$$\boldsymbol{A}=\boldsymbol{D}=\mathbf{g},\qquad \boldsymbol{B}=\mathbf{g}\begin{pmatrix}\frac{\lambda}{2}\tau_0 & 0\\ 0 & \boldsymbol{\tau}\end{pmatrix},\quad \boldsymbol{C}=\mathbf{g}\begin{pmatrix}\tau_0 & 0\\ 0 & \mathbf{0}\end{pmatrix},$$

with $\boldsymbol{\tau}$ as above and $\mathbf{0}$ the 2×2 zero matrix. Finally, the analogue of (3.4) yields,

$$\epsilon''=\epsilon'\epsilon\left(1-\frac{\lambda}{2}\tau_0'\tau_0\right),\quad \mathbf{a}''=\mathbf{a}'\mathbf{a},\quad \tau_0''=\tau_0'+\tau_0,\quad \boldsymbol{\tau}''=\boldsymbol{\tau}+\mathbf{a}^{-1}\boldsymbol{\tau}'\mathbf{a}, \tag{3.7}$$

where

$$\mathbf{a}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

is an element of SL_2 and similar expressions for $\mathbf{a}''$ and $\mathbf{a}'$. Note that this is again of the form (1.1) with,

$$\mathbf{g} = \begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} \tau_0 & 0 \\ 0 & \boldsymbol{\tau} \end{pmatrix},$$

with

$$\gamma_0 = \begin{pmatrix} -\dfrac{\lambda}{2}\tau_0'\tau_0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} \qquad \text{and} \qquad \gamma_1 = 0 \tag{3.8}$$

The identity element is

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} \right),$$

where $\mathbb{1}$ and $\mathbf{0}$ are the 2×2 identity and zero matrices, respectively. Similarly, the inverse element of $(\mathbf{g}, \gamma)$ is given by

$$(\mathbf{g}, \gamma)^{-1} = \left(\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \mathbf{a}^{-1} \end{pmatrix}, \begin{pmatrix} -\tau_0 & 0 \\ 0 & -\mathbf{a}\boldsymbol{\tau}\mathbf{a}^{-1} \end{pmatrix} \right).$$

Remark. A word must be said about the Lie supergroups based on GL_2 associated to the adjoint representation, and having $\nu \neq 0$. It turns out that the general formula for the corresponding multiplication law depending on arbitrary parameter values of λ, μ, and ν is awkward and not particularly illuminating. However, the generic Lie supergroup having $\lambda\mu\nu \neq 0$ is isomorphic to one having a particularly simple multiplication map; namely, $\mathrm{GL}_2(\mathbb{F}; 2, 2, 1)$. This will be worked out in full in the next section (see Theorem 4.1).

4. Abstract form for the multiplication morphisms and commutative-diagram characterization of left-invariant supervector fields

We want to determine the left-invariant supervector fields for each multiplication law we have found. In order to do that, we first have to know what conditions must such supervector fields satisfy in a coordinate-free manner, and encapsulate that information inside some appropriate commutative diagram. According to the Lie supergroups theory, every Lie supergroup $(G, \mathcal{A}_G)$ comes equipped with a special morphism that plays the role of the identity element $\varepsilon : (G, \mathcal{A}_G) \to (G, \mathcal{A}_G)$ such that $m \circ (\mathrm{i}d, \varepsilon) = \mathrm{i}d = m \circ (\varepsilon, \mathrm{i}d)$ (see [1]).

Let $(G, \mathcal{A}_G)$ be a Lie supergroup and let X be a supervector field in $(G, \mathcal{A}_G)$, i.e., $X \in \mathrm{Der}_{\mathcal{A}_G}(G)$. Define $\widehat{X}$ as the unique element in $\mathrm{Der}_{\mathcal{A}_{G\times G}}(G \times G)$ that satisfies the conditions $\widehat{X} p_1^* f = 0$ and $\widehat{X} p_2^* f = p_2^* X f$, for every $f \in \mathcal{A}_G$, where $p_i : (G, \mathcal{A}_G) \times (G, \mathcal{A}_G) \to$

$(G, \mathcal{A}_G)$ stands for the appropriate projections onto the ith factor. Define the map $\varepsilon^{(2)}$: $(G, \mathcal{A}_G) \to (G, \mathcal{A}_G) \times (G, \mathcal{A}_G)$ by $\varepsilon^{(2)^*} p_1^* = \mathrm{i}d^*$ and $\varepsilon^{(2)^*} p_2^* = \varepsilon^*$. So, X is a left-invariant vector field if the diagram

$$\begin{tikzcd}
\mathcal{A}_{G\times G} \arrow[r, "(p_1,\mathrm{m})^*"] \arrow[d, "\widehat{X}"'] & \mathcal{A}_{G\times G} \arrow[r, "\widehat{X}"] & \mathcal{A}_{G\times G} \arrow[d, "\varepsilon^{(2)^*}"] \\
\mathcal{A}_{G\times G} \arrow[r, "(p_1,\mathrm{m})^*"'] & \mathcal{A}_{G\times G} \arrow[r, "\varepsilon^{(2)^*}"'] & \mathcal{A}_G
\end{tikzcd} \tag{4.1}$$

commutes, i.e., if $\varepsilon^{(2)^*} \circ \widehat{X} \circ (p_1, m)^* = \varepsilon^{(2)^*} \circ (p_1, m)^* \circ \hat{X}$. We mention in passing that an alternative approach to the definition of left-invariant supervector fields can be accomplished via the use of appropriate functors as has been done by Varadarajan in [11].

The morphism m associated to the multiplication law (3.5) has the following effect on the local coordinates:

$$\begin{aligned}
m^* x_{ij} &= \sum_{k=1}^{2} p_1^* x_{ik} p_2^* x_{kj} + \left(\mu - \frac{\lambda}{2}\right) \left(\frac{p_1^*\xi_{11} + p_1^*\xi_{22}}{2}\right) \left(\frac{p_2^*\xi_{11} + p_2^*\xi_{22}}{2}\right) \\
&\quad \times \sum_{k=1}^{2} p_1^* x_{ik} p_2^* x_{kj} + \mu \left(\frac{p_2^*\xi_{11} + p_2^*\xi_{22}}{2}\right) \sum_{k,\ell=1}^{2} p_1^* x_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j}, \\
m^* \xi_{ij} &= p_2^* \xi_{ij} + \sum_{k,\ell=1}^{2} p_2^* u_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j} - \mu \left(\frac{p_2^*\xi_{11} + p_2^*\xi_{22}}{2}\right) \\
&\quad \times \left(\sum_{k,\ell=1}^{2} p_2^* u_{ik} p_1^* \xi_{k\ell} p_2^* x_{\ell j}\right)^2,
\end{aligned}$$

where, x_{ij} and ξ_{ij} are the projection maps defined by $x_{ij}(\mathbf{g}, \gamma) = \mathbf{g}_{ij}$ and $\xi_{ij}(\mathbf{g}, \gamma) = \gamma_{ij}$ and

$$(u_{ij}) = (x_{11}x_{22} - x_{12}x_{21})^{-1} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \text{ for } i, j \in \{1, 2\}.$$

It also is easy to see that $\varepsilon^* x_{ij} = \delta_{ij}$, $\varepsilon^* \xi_{ij} = 0$, and, $\varepsilon^* f = \tilde{f}(\mathbb{1})$, for every $f \in \mathcal{A}_{\mathrm{GL}_2(\mathbb{C};\lambda,\mu,\nu=0)}$, where $\mathbb{1}$ is the 2×2 identity matrix, and $f \mapsto \tilde{f}$ stands for the forgetful functor $\mathcal{A}_{\mathrm{GL}_2(\mathbb{C};\lambda,\mu=\nu=0)} \to C^\infty_{\mathrm{GL}_2}$.

Note that if the local expression of a supervector field X is written as

$$X = \sum_{m,n=1}^{2} A_{mn} \frac{\partial}{\partial x_{mn}} + B_{mn} \frac{\partial}{\partial \xi_{mn}}$$

the local expression for $\hat{X}$ is,

$$\hat{X} = \sum_{m,n=1}^{2} p_2^* A_{mn} \frac{\partial}{\partial p_2^* x_{mn}} + p_2^* B_{mn} \frac{\partial}{\partial p_2^* \xi_{mn}}.$$

A straightforward computation from (4.1) shows that X is a left-invariant supervector field for the Lie supergroup structure given in (3.5) if and only if,

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{2} x_{ik} \tilde{A}_{kj}(\mathbb{1}) + \mu \left(\frac{\tilde{B}_{11}(\mathbb{1}) + \tilde{B}_{22}(\mathbb{1})}{2} \right) x_{ik} \xi_{kj} \\ &\quad - \left(\mu - \frac{\lambda}{2} \right) \left(\frac{\tilde{B}_{11}(\mathbb{1}) + \tilde{B}_{22}(\mathbb{1})}{2} \right) \times \left(\frac{\xi_{11} + \xi_{22}}{2} \right) x_{ij}, \\ B_{ij} &= \tilde{B}_{ij}(\mathbb{1}) + \sum_{k=1}^{2} \xi_{ik} \tilde{A}_{kj}(\mathbb{1}) - \tilde{A}_{ik}(\mathbb{1}) \xi_{kj} - \mu \left(\frac{\tilde{B}_{11}(\mathbb{1}) + \tilde{B}_{22}(\mathbb{1})}{2} \right) \xi_{ik} \xi_{kj} \end{aligned}$$

and therefore, we can write $X = \sum_{p,q=1}^{2} \tilde{A}_{pq}(\mathbb{1}) X_{pq} + \tilde{B}_{pq}(\mathbb{1}) Y_{pq}$, where

$$\begin{aligned} X_{pq} &= \sum_{k=1}^{2} x_{kp} \frac{\partial}{\partial x_{kq}} + \xi_{kp} \frac{\partial}{\partial \xi_{kq}} - \xi_{qk} \frac{\partial}{\partial \xi_{pk}}, \\ Y_{pq} &= \frac{\partial}{\partial \xi_{pq}} + \frac{\delta_{pq}}{2} \left\{ \sum_{i,j=1}^{2} \left(\left(\frac{\lambda}{2} - \mu \right) \left(\frac{\xi_{11} + \xi_{22}}{2} \right) x_{ij} + \mu \sum_{k=1}^{2} x_{ik} \xi_{kj} \right) \frac{\partial}{\partial x_{ij}} \right. \\ &\quad \left. - \mu \sum_{i,j,k=1}^{2} \xi_{ik} \xi_{kj} \frac{\partial}{\partial \xi_{ij}} \right\}. \end{aligned}$$

Moreover,

$$[X_{pq}, X_{rs}] = \delta_{rq} X_{ps} - \delta_{ps} X_{rq} \quad \text{and} \quad [X_{pq}, Y_{rs}] = \delta_{rq} Y_{ps} - \delta_{ps} Y_{rq}.$$

Thus, putting

$$\begin{aligned} &x_0 = X_{11} + X_{22}, \qquad x_1 = X_{11} - X_{22}, \qquad x_2 = X_{12}, \qquad x_3 = X_{21}, \\ &y_0 = Y_{11} + Y_{22}, \qquad y_1 = Y_{11} - Y_{22}, \qquad y_2 = Y_{12} \qquad y_3 = Y_{21}, \end{aligned}$$

we recover the Lie superalgebras $\mathfrak{gl}_2(\mathbb{F}; \lambda, \mu, \nu = 0)$ admitting a faithful representation in $\mathbb{F}^{2|2}$ according to Lemma 2.1.

Similarly, we may now compute, the left-invariant supervector fields for the Lie supergroups whose multiplication law is (3.7). One may compute m^* from (3.7), and write $X = \sum_{m,n} A_{mn} \frac{\partial}{\partial x_{mn}} + B_{mn} \frac{\partial}{\partial \xi_{mn}}$ as before. A straightforward computation shows that X is a left-invariant supervector field if and only if

$$X = \tilde{A}_{00}(\mathbb{1})X_{00} + \tilde{A}_{ij}(\mathbb{1})X_{ij} + \tilde{B}_{00}(\mathbb{1})Y_{00} + \tilde{B}_{ij}(\mathbb{1})Y_{ij},$$

where

$$X_{00} = x_{00}\frac{\partial}{\partial x_{00}}, \qquad X_{ij} = \sum_{k=1}^{2} x_{ki}\frac{\partial}{\partial x_{kj}} + \xi_{ki}\frac{\partial}{\partial \xi_{kj}} - \xi_{jk}\frac{\partial}{\partial \xi_{ik}},$$

$$Y_{00} = \frac{\partial}{\partial \xi_{00}} - \frac{\lambda}{2}x_{00}\xi_{00}\frac{\partial}{\partial x_{00}}, \qquad Y_{ij} = \frac{\partial}{\partial \xi_{ij}}.$$

Finally, it is also a straightforward matter to verify that the Lie superalgebra equivalence class corresponding to $[\lambda \neq 0, \mu = 0, \nu = 0]$ (i.e. cases in (4) of Lemma 2.1) can be faithfully realized.

Our next result deals with the multiplication law for the Lie supergroup $\mathrm{GL}_2(\mathbb{F}; 2, 2, 1)$ for which all the Lie supergroups having $\lambda\mu\nu \neq 0$ are isomorphic to.

Theorem 4.1. *Let $\mathbb{C}$ be the ground field, $\gamma_{11}, \gamma_{12}, \gamma_{21}$ and γ_{22} be odd elements and let $\mathrm{GL}_2(\mathbb{C}; 2, 2, 1)$ be the group of 2×2 matrices with entries in $\wedge[\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]$—the exterior algebra generated by $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$. Let $\mathbf{g} + \gamma$ be an element in $\mathrm{GL}_2(\mathbb{C}; 2, 2, 1)$, and let x_{ij} and ξ_{ij} be the projection maps defined by $x_{ij}(\mathbf{g} + \gamma) = \mathbf{g}_{ij}$ and $\xi_{ij}(\mathbf{g} + \gamma) = \gamma_{ij}$. Define a multiplication law in $\mathrm{GL}_2(\mathbb{C}; 2, 2, 1)$ by*

$$m^*(x_{ij}) = \sum_{k=1}^{2} p_1^*(x_{ik})p_2^*(x_{kj}) + (-1)^{i+k}p_1^*(\xi_{ik})p_2^*(\xi_{kj}),$$

$$m^*(\xi_{ij}) = \sum_{k=1}^{2} (-1)^{i+k}p_1^*(x_{ik})p_2^*(\xi_{kj}) + p_1^*(\xi_{ik})p_2^*(x_{kj}).$$

The left-invariant supervector fields associated to this multiplication are

$$X_{pq} = \sum_{k=1}^{2} x_{kp}\frac{\partial}{\partial x_{kq}} + \xi_{kp}\frac{\partial}{\partial \xi_{kq}}, \qquad Y_{pq} = \sum_{k=1}^{2} (-1)^k x_{kp}\frac{\partial}{\partial \xi_{kq}} + (-1)^{k+1}\xi_{kp}\frac{\partial}{\partial x_{kq}}$$

satisfying

$$[X_{pq}, X_{rs}] = \delta_{rq}X_{ps} - \delta_{ps}X_{rq}, \qquad [X_{pq}, Y_{rs}] = \delta_{rq}Y_{ps} - \delta_{ps}Y_{rq},$$

$$[Y_{pq}, Y_{rs}] = -\delta_{rq}X_{ps} - \delta_{ps}X_{rq}$$

and, by setting

$$x_0 = -X_{11} - X_{22}, \qquad x_1 = -X_{11} + X_{22}, \qquad x_2 = -X_{12}, \qquad x_3 = -X_{21},$$

$$y_0 = Y_{11} + Y_{22}, \qquad y_1 = Y_{11} - Y_{22}, \qquad y_2 = Y_{12}, \qquad y_3 = Y_{21},$$

we recover the Lie superalgebra associated to the parameters $\lambda = \mu = 2$ and $\nu = 1$.

Proof. It is a straightforward matter to check that the given multiplication morphism is associative. The identity morphism id is given by

$$\mathrm{i}d^* x_{ij} = x_{ij} \quad \text{and} \quad \mathrm{i}d^* \xi_{ij} = \xi_{ij},$$

whereas the inversion morphism α is given by

$$\alpha^*(x_{ij} + \xi_{ij}) = \mathbf{y}_{ij} - ((\mathbf{y}\xi)\mathbf{y})_{ij} + ((\mathbf{y}\xi)^2\mathbf{y})_{ij} - ((\mathbf{y}\xi)^3\mathbf{y})_{ij} + ((\mathbf{y}\xi)^4\mathbf{y})_{ij},$$

where

$$\mathbf{y} = (\mathbf{y}_{ij}) = (x_{11}x_{22} - x_{12}x_{21})^{-1} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \quad \text{and} \quad \xi = (\xi_{ij}).$$

Using the same techniques as above, we prove that $\varepsilon^* x_{ij} = \delta_{ij}$, $\varepsilon^*\xi_{ij} = 0$, $\varepsilon^* f = \tilde{f}(\mathbb{1})$ and actually, if $X = \sum_{m,n=1}^{2} A_{mn} \frac{\partial}{\partial x_{mn}} + B_{mn} \frac{\partial}{\partial \xi_{mn}}$ is a supervector field, then X is a left-invariant supervector field if

$$X = \sum_{i,j=1}^{2} \tilde{A}_{ij}(\mathbb{1}) X_{ij} + \tilde{B}_{ij}(\mathbb{1}) Y_{ij},$$

where

$$X_{ij} = \sum_{k=1}^{2} x_{ki} \frac{\partial}{\partial x_{kj}} + \xi_{ki} \frac{\partial}{\partial \xi_{kj}}, \quad Y_{ij} = (-1)^i \left(\sum_{k=1}^{2} (-1)^{k+1} \xi_{ki} \frac{\partial}{\partial x_{kj}} + (-1)^k x_{ki} \frac{\partial}{\partial \xi_{kj}} \right),$$

where $(-1)^i$ appearing in Y_{ij} can be included in $\tilde{B}_{ij}$. We have therefore obtained a faithful representation for the equivalence class $\lambda = 2$, $\mu = 2$, $\nu = 1$. □

Remark. The multiplication law given in this proposition was taken from [10]. It has been shown there that the special form of this matrix product, actually corresponds to the composition law for two endomorphisms on the graded vector space of dimension (2, 2) (see also other references by the same author in [10]). Note that the supergroup defined by this multiplication law has sometimes appeared in the literature under the name $Q(2)$.

The multiplication law given in Theorem 4.1 is *generic* in the following sense: the set $\dot{\mathbb{C}}^3 = (\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})$ is an open set on $\mathbb{C}^3$. According to Theorem 1.1, the Lie superalgebra generated by one element (λ, μ, ν) of this open set is isomorphic to the Lie superalgebra represented by the selections $\lambda' = 2$, $\mu' = 2$ and $\nu' = 1$ and then, the multiplication law of the Lie supergroup associate to the parameters λ, μ, ν is isomorphic to the one stated in Theorem 4.1.

5. Compact real forms

Let us consider the real Lie superalgebras $\mathfrak{u}_2(\lambda, \mu, \nu)$ with underlying Lie algebra $\mathfrak{u}_2$, that arise after changing the basis in $\mathfrak{gl}_2(\mathbb{C}; \lambda, \mu, \nu)$ by $w_0 = \mathrm{i}I$, $w_3 = \mathrm{i}H$, $w_2 = E - F$ and $w_1 = \mathrm{i}(E + F)$, as usual. By letting P as before, a change of parity map, we have that the symmetric bilinear equivariant map $\Phi : \mathfrak{u}_2 \times \mathfrak{u}_2 \to \mathfrak{u}_2$ that gives the Lie bracket for any pair of odd elements, where $\Phi(z, w) = [P(z), P(w)]$, is

$$\begin{aligned}
&\Phi(w_0, w_0) = \mathrm{i}\lambda w_0, &&\Phi(w_0, w_3) = \mathrm{i}\mu w_3, &&\Phi(w_3, w_3) = 2\mathrm{i}\nu w_0,\\
&\Phi(w_0, w_2) = \mathrm{i}\mu w_2, &&\Phi(w_3, w_2) = 0, &&\Phi(w_2, w_2) = 2\mathrm{i}\nu w_0,\\
&\Phi(w_0, w_1) = \mathrm{i}\mu w_1, &&\Phi(w_3, w_1) = 0, &&\Phi(w_2, w_1) = 0,\\
&\Phi(w_1, w_1) = 2\mathrm{i}\nu w_0.
\end{aligned}$$

Then, in order to have the compact real form for GL_2, λ, μ and ν have to be restricted so as to be purely imaginary.

As in Section 2, we have faithful representations for all these Lie superalgebras in supervector fields of the supermanifolds $\mathbb{F}^{2|2}$ and $\mathbb{F}^{3|3}$: one only needs to note that the generators are now W_k (with $|W_k| = 0$) and Z_k (with $|Z_k| = 1$), where $W_k = \mathrm{i}X_k$ and $Z_k = \mathrm{i}Y_k$ for $k \in \{0, 1, 3\}$ whereas $W_2 = X_2$ and $Z_2 = Y_2$. So, proceeding as in Section 3, we verify that if $\Gamma^*_{\mathbf{g}} = \mathrm{Exp}(t_0 W_0) \circ \mathrm{Exp}(t_3 W_3) \circ \mathrm{Exp}(t_2 W_2) \circ \mathrm{Exp}(t_1 W_1)$, we obtain

$$\Gamma^*_{\mathbf{g}} : \begin{cases} z^1 \mapsto \alpha z^1 + \gamma z^2 \\ z^2 \mapsto \beta z^1 + \delta z^2 \\ \zeta^1 \mapsto \alpha \zeta^1 + \gamma \zeta^2 \\ \zeta^2 \mapsto \beta \zeta^1 + \delta \zeta^2, \end{cases}$$

where

$$\begin{aligned}
\alpha &= (\cos t_1 \cos t_2 + \mathrm{i} \sin t_1 \sin t_2)\, \mathrm{e}^{\mathrm{i}(t_0 + t_3)},\\
\beta &= (\mathrm{i} \sin t_1 \cos t_2 + \cos t_1 \sin t_2)\, \mathrm{e}^{\mathrm{i}(t_0 + t_3)},\\
\gamma &= (-\cos t_1 \sin t_2 + \mathrm{i} \sin t_1 \cos t_2)\, \mathrm{e}^{\mathrm{i}(t_0 - t_3)},\\
\delta &= (\cos t_1 \cos t_2 - \mathrm{i} \sin t_1 \sin t_2)\, \mathrm{e}^{\mathrm{i}(t_0 - t_3)}.
\end{aligned}$$

We therefore see that, up to $\mathrm{e}^{\mathrm{i}t_0}$, $\delta = \overline{\alpha}$ and $\gamma = -\overline{\beta}$. In other words, the underlying Lie group is U_2, as expected. On the other hand,

$$\Gamma^*_{Z_0} = \mathrm{Exp}(\tau_0 Z_0) : \begin{cases} z^1 \mapsto z^1 + \mathrm{i}k\tau_0 \zeta^1 \\ z^2 \mapsto z^2 + \mathrm{i}k\tau_0 \zeta^2 \\ \zeta^1 \mapsto \zeta^1 + \mathrm{i}g\tau_0 z^1 \\ \zeta^2 \mapsto \zeta^2 + \mathrm{i}g\tau_0 z^2 \end{cases},$$

$$\Gamma^*_{Z_3} = \mathrm{Exp}(\tau_3 Z_3) : \begin{cases} z^1 \mapsto z^1 + \mathrm{i}e\tau_3\zeta^1 \\ z^2 \mapsto z^2 - \mathrm{i}e\tau_3\zeta^2 \\ \zeta^1 \mapsto \zeta^1 + \mathrm{i}d\tau_3 z^1 \\ \zeta^2 \mapsto \zeta^2 - \mathrm{i}d\tau_3 z^2 \end{cases},$$

$$\Gamma^*_{Z_2} = \mathrm{Exp}(\tau_2 Z_2) : \begin{cases} z^1 \mapsto z^1 - e\tau_2\zeta^2 \\ z^2 \mapsto z^2 + e\tau_2\zeta^1 \\ \zeta^1 \mapsto \zeta^1 - d\tau_2 z^2 \\ \zeta^2 \mapsto \zeta^2 + d\tau_2 z^1 \end{cases},$$

$$\Gamma^*_{Z_1} = \mathrm{Exp}(\tau_1 Z_1) : \begin{cases} z^1 \mapsto z^1 + \mathrm{i}e\tau_1\zeta^2 \\ z^2 \mapsto z^2 + \mathrm{i}e\tau_1\zeta^1 \\ \zeta^1 \mapsto \zeta^1 + \mathrm{i}d\tau_1 z^2 \\ \zeta^2 \mapsto \zeta^2 + \mathrm{i}d\tau_1 z^1 \end{cases}$$

and, from $\Psi(\mathbf{g}, \tau_0.\tau_1, \tau_2, \tau_3) := \Gamma^*_{\mathbf{g}} \circ \mathrm{Exp}(\tau_0 Z_0) \circ \mathrm{Exp}(\tau_3 Z_3) \circ \mathrm{Exp}(\tau_2 Z_2) \circ \mathrm{Exp}(\tau_1 Z_1)$, we already know that

$$\Psi\mathbf{z} = \boldsymbol{A}\mathbf{z} + \boldsymbol{C}\zeta \quad \text{and} \quad \Psi\zeta = \boldsymbol{B}\mathbf{z} + \boldsymbol{D}\zeta,$$

where we write Ψ, $\mathbf{z}$ and ζ as a shorthand notation as in Section 3. Once more, the cases when $\nu = 0$ are simple to compute: setting $e = 0$ we deduce from (3.4) the following expression for the product $(\mathbf{g}', \mathrm{i}\tau_0', \mathrm{i}\boldsymbol{\tau}') \cdot (\mathbf{g}, \mathrm{i}\tau_0, \mathrm{i}\boldsymbol{\tau})$:

$$\left(\mathbf{g}'\mathbf{g} - \frac{\lambda}{2}\mathrm{i}\tau_0'\mathrm{i}\tau_0\mathbf{g}'\mathbf{g} + \mu\mathrm{i}\tau_0\mathbf{g}'\mathrm{i}\boldsymbol{\tau}'\mathbf{g},\ \mathrm{i}\tau_0' + \mathrm{i}\tau_0, \mathrm{i}\boldsymbol{\tau} + \mathbf{g}^{-1}\mathrm{i}\boldsymbol{\tau}'\mathbf{g} - \mu\mathrm{i}\tau_0(\mathbf{g}^{-1}\mathrm{i}\boldsymbol{\tau}'\mathbf{g})^2\right), \tag{5.1}$$

where

$$\mathrm{i}\boldsymbol{\tau}' = \begin{pmatrix} \mathrm{i}\tau_3' & \mathrm{i}\tau_1' + \tau_2' \\ \mathrm{i}\tau_1' - \tau_2' & -\mathrm{i}\tau_3' \end{pmatrix} \text{ and } \mathrm{i}\boldsymbol{\tau} = \begin{pmatrix} \mathrm{i}\tau_3 & \mathrm{i}\tau_1 + \tau_2 \\ \mathrm{i}\tau_1 - \tau_2 & -\mathrm{i}\tau_3 \end{pmatrix}.$$

Simple computations show that left-invariant supervector fields associated to this multiplication law are easily found as in Section 4 for the equivalence class of $[\lambda, \mu, \nu = 0]$ represented in $\mathbb{F}^{2|2}$.

For the case $\lambda \neq 0$, $\mu = 0$, $\nu = 0$ arising from the class represented in $\mathbb{F}^{3|3}$ as stated in (4) of Lemma 2.1, we obtain the following multiplication law for the

$$(\mathbf{g}', \gamma') = \left(\begin{pmatrix} \epsilon' & 0 \\ 0 & \mathbf{a}' \end{pmatrix}, \begin{pmatrix} \mathrm{i}\tau_0' & 0 \\ 0 & \mathrm{i}\boldsymbol{\tau}' \end{pmatrix}\right) \text{ and } (\mathbf{g}, \gamma) = \left(\begin{pmatrix} \epsilon & 0 \\ 0 & \mathbf{a} \end{pmatrix}, \begin{pmatrix} \mathrm{i}\tau_0 & 0 \\ 0 & \mathrm{i}\boldsymbol{\tau} \end{pmatrix}\right)$$

elements:

$$(\mathbf{g}', \gamma') \cdot (\mathbf{g}, \gamma) = \left(\begin{pmatrix} \epsilon'\epsilon \left(1 - \frac{\lambda}{2}\mathrm{i}\tau_0'\mathrm{i}\tau_0\right) & 0 \\ 0 & \mathbf{a}'\mathbf{a} \end{pmatrix}, \begin{pmatrix} \mathrm{i}\tau_0' + \mathrm{i}\tau_0 & 0 \\ 0 & \mathrm{i}\boldsymbol{\tau} + \mathbf{a}^{-1}\mathrm{i}\boldsymbol{\tau}'\mathbf{a} \end{pmatrix} \right), \tag{5.2}$$

where

$$\mathrm{i}\boldsymbol{\tau}' = \begin{pmatrix} \mathrm{i}\tau_3' & \mathrm{i}\tau_1' + \tau_2' \\ \mathrm{i}\tau_1' - \tau_2' & -\mathrm{i}\tau_3' \end{pmatrix} \text{ and } \mathrm{i}\boldsymbol{\tau} = \begin{pmatrix} \mathrm{i}\tau_3 & \mathrm{i}\tau_1 + \tau_2 \\ \mathrm{i}\tau_1 - \tau_2 & -\mathrm{i}\tau_3 \end{pmatrix}.$$

Once more, simple computations shows that the left-invariant supervector fields associated to this multiplication morphism (see Section 3) brings us back to the Lie superalgebra we started with in the equivalence class of $[\lambda \neq 0, \mu = 0, \nu = 0]$.

6. Maximal torus and supertori associated to the adjoint representation

From our results in Section 5 we know that

$$\Phi(w_0, w_0) = \mathrm{i}\lambda w_0, \qquad \Phi(w_0, w_3) = \mathrm{i}\mu w_3, \qquad \Phi(w_3, w_3) = 2\mathrm{i}\nu w_0$$

and we have realizations in supervector fields in $\mathbb{R}^{2|2}$ and $\mathbb{R}^{3|3}$ supermanifolds given by the appropriate restrictions. We now want to compute a general composition law in terms of the arbitrary parameter values $[\lambda, \mu, \nu]$.

Proposition 6.1. *Lie superalgebras in the equivalence classes* $[\lambda, \mu, \nu]$ *admit faithful representations in terms of supervector fields in the supermanifold* $\mathbb{R}^{2|2}$ *with local coordinates* $\{z^1, z^2; \zeta^1, \zeta^2\}$ *given by*

$$W_0 = \mathrm{i}\left(z^1\frac{\partial}{\partial z^1} + z^2\frac{\partial}{\partial z^2} + \zeta^1\frac{\partial}{\partial \zeta^1} + \zeta^2\frac{\partial}{\partial \zeta^2}\right),$$
$$W_3 = \mathrm{i}\left(z^1\frac{\partial}{\partial z^1} - z^2\frac{\partial}{\partial z^2} + \zeta^1\frac{\partial}{\partial \zeta^1} - \zeta^2\frac{\partial}{\partial \zeta^2}\right),$$
$$Z_0 = \mathrm{i}k\left(\zeta^1\frac{\partial}{\partial z^1} + \zeta^2\frac{\partial}{\partial z^2}\right) + \mathrm{i}g\left(z^1\frac{\partial}{\partial \zeta^1} + z^2\frac{\partial}{\partial \zeta^2}\right),$$
$$Z_3 = \mathrm{i}e\left(\zeta^1\frac{\partial}{\partial z^1} - \zeta^2\frac{\partial}{\partial z^2}\right) + \mathrm{i}d\left(z^1\frac{\partial}{\partial \zeta^1} - z^2\frac{\partial}{\partial \zeta^2}\right),$$

where $\lambda = 2gk$, $\mu = eg + dk$ *and* $\nu = ed$.

The integral flows for these supervector fields can be computed as before and it is not difficult to see that the multiplication law for the elements $(\mathbf{g}', \mathrm{i}\boldsymbol{\tau}'_0, \mathrm{i}\boldsymbol{\tau}'_3)$ and $(\mathbf{g}, \mathrm{i}\boldsymbol{\tau}_0, \mathrm{i}\boldsymbol{\tau}_3)$ is

$$\left(\mathbf{g}'\mathbf{g}\left\{\left(\mathbb{1}-\frac{\lambda}{2}\mathrm{i}\boldsymbol{\tau}'_0\mathrm{i}\boldsymbol{\tau}_0\right)(\mathbb{1}-\nu\mathrm{i}\boldsymbol{\tau}'_3\mathrm{i}\boldsymbol{\tau}_3)+\mu\mathrm{i}\boldsymbol{\tau}'_0\mathrm{i}\boldsymbol{\tau}_3\right\}, \mathrm{i}\boldsymbol{\tau}'_0+\mathrm{i}\boldsymbol{\tau}_0, \mathrm{i}\boldsymbol{\tau}'_3+\mathrm{i}\boldsymbol{\tau}_3\right). \qquad (6.1)$$

This multiplication law exhibits the λ, μ, ν parameters in general. The left-invariant supervector fields can be computed as in Section 4 and it is a straightforward matter to prove that they bring us back to the (2, 2)-dimensional toral superalgebras we started with.

There is a related problem to understand within the spirit that has guided us throughout this work: namely, *to classify all Lie superalgebras whose underlying 2-dimensional Lie algebra is Abelian under the assumption that the action of the even Lie algebra into the odd module is given via the adjoint representation.* These Lie superalgebras are classified by symmetric bilinear maps $\Phi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ with no restrictions, since the Jacobi identities are trivially satisfied.

Let $\mathfrak{g}_0 = \mathrm{Span}_{\mathbb{R}}\{w_1, w_2\}$ be the Abelian 2-dimensional Lie algebra and let $\mathfrak{g}_1 = \{Pw_1, Pw_2\}$ be the $\mathfrak{g}_0$-module defined by the adjoint representation. Then

$$\Phi(Pw_j, Pw_j) = \theta^1_{ij}w_1 + \theta^2_{ij}w_2$$

defines a Lie superalgebra structure for arbitrary parameters θ^k_{ij} in $\mathbb{R}$. A different symmetric bilinear map $\Phi' : \mathfrak{g}'_1 \times \mathfrak{g}'_1 \to \mathfrak{g}'_0$ would yield a different set of parameters $(\theta')^k_{ij}$. The Lie superalgebras generated by θ^k and $(\theta')^k$ will be isomorphic if and only if there is a Lie algebra isomorphism $T : \mathfrak{g}_0 \to \mathfrak{g}'_0$ of the Abelian Lie algebra (actually, any linear isomorphism $T \in \mathrm{GL}_2$ will do it) and a linear isomorphism $S : \mathfrak{g}_1 \to \mathfrak{g}'_1$ such that $\Phi'(S(x), S(y)) = T(\Phi(x, y))$ for any $x, y \in \mathfrak{g}_1$. This condition can be written in terms of matrices as

$$S^{\mathrm{t}}(\theta')^1S = T_{11}\theta^1 + T_{12}\theta^2, \qquad S^{\mathrm{t}}(\theta')^2S = T_{21}\theta^1 + T_{22}\theta^2. \qquad (6.2)$$

Type	$\tilde{\theta}^1$	$\tilde{\theta}^2$
1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
5	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
6	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
7	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Type	$[\lambda,\ \mu,\ \nu]$
1	$[0, 0, 0]$
2	$[0, 0, 1]$, $[1, 0, 0]$
3	$[1, 0, 1]$
4	$[1, 1, 1]$
5	$[1, 0, -1]$, $[0, 1, 0]$
6	$[1, 1, -1]$
7	$[0, 1, 1]$, $[1, 1, 0]$

Therefore, we can approach the corresponding classification problem, whose solution is stated in the following proposition. Its corollary, on the other hand, shows what the relationship is between the maximal tori found in the last section and the supertori given by the classification problem just posed.

Proposition 6.2. *The group* $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ *acts on the left of* $\mathrm{Sym}_2(\mathbb{R}) \times \mathrm{Sym}_2(\mathbb{R})$ *via*

$$(T, S) \cdot (\theta^1, \theta^2) = (T_{11}S \cdot \theta^1 + T_{12}S \cdot \theta^2,\ T_{21}S \cdot \theta^1 + T_{22}S \cdot \theta^2),$$

where $\mathrm{Sym}_2(\mathbb{R})$ *is the set of symmetric* 2×2 *matrices over* $\mathbb{R}$ *and* $S \cdot \theta = S^{-1}\theta(S^{-1})^{\mathrm{t}}$ *is the left action of* $\mathrm{GL}_2(\mathbb{R})$ *on* $\mathrm{Sym}_2(\mathbb{R})$ *that results from Eq.* (6.2) *above*. *This action defines seven different orbits whose representatives* $\tilde{\theta}^1$ *and* $\tilde{\theta}^2$ *are listed in the following table*:

Corollary 6.1. *There is a surjection from the set of maximal tori in Proposition* 6.1, *onto the set of tori obtained from the action just defined*.

Proof of corollary. For real cases in λ, μ and ν, we know that

$$\theta^1 = \begin{pmatrix} \lambda & 0 \\ 0 & \nu \end{pmatrix} \text{ and } \theta^2 = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}.$$

We can see these cases in terms of the above Type as follows:

Proof of proposition. Let us explain what the philosophy of the proof is. By means of the action $(\theta^1, \theta^2) \mapsto ((S^{-1})^{\mathrm{t}}\theta^1 S^{-1}, (S^{-1})^{\mathrm{t}}\theta^2 S^{-1})$, we try first to see under what conditions can both $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ and $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ be brought to a diagonal form. Once they are both diagonal, we can further act with an appropriate group element $T \in \mathrm{GL}_2(\mathbb{R})$ so as to simplify each $\tilde{\theta}^i = T_{i1}(S^{-1})^{\mathrm{t}}\theta^1 S^{-1} + T_{i2}(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ $(i = 1, 2)$ as much as possible. There are some cases in which it is impossible to simultaneously have $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ and $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ in diagonal form. These cases are then treated separately. At the end, one only needs to check that with the chosen representatives one really reaches any pair of symmetric matrices under the given $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$-action and that the representatives really belong to different orbits. □

There are a few simple cases where one immediately knows that both, θ^1 and θ^2, can be simultaneously diagonalized. Say, if from the outset, θ^1 is proportional to θ^2, then both can be diagonalized at once with the same $S \in \mathrm{GL}_2(\mathbb{R})$. If this is the case (say $\theta^2 = a\theta^1$, with $a \neq 0$), several subcases have to be considered: Namely, either θ^1 is positive definite; or θ^1 is negative definite; or θ^1 is nondegenerate but nondefinite; or θ^1 has rank-one with a positive eigenvalue; or θ^1 has rank-one with a negative eigenvalue; or θ^1 is identically zero.

In all these cases, by choosing an appropriate $T \in \mathrm{GL}_2(\mathbb{R})$ one can easily see that if the eigenvalues of θ^1 either have equal signs, or one of them is zero, then $\tilde{\theta}^1 = T_{11}(S^{-1})^{\mathrm{t}}\theta^1 S^{-1} + T_{12}(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ can be chosen so as to be either the identity matrix, or the diagonal matrix with diagonal entries $(1, 0)$ if θ^1 was rank-one, or diagonal entries $(0, 0)$ if θ^1 was identically zero. In any case, the choice of T can also be adjusted so as to have $\tilde{\theta}^2 = T_{21}(S^{-1})^{\mathrm{t}}\theta^1 S^{-1} + T_{22}(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ identically zero. This accounts for the first three types in the statement of the proposition, plus Type 5.

There are other less obvious cases where one can simultaneously diagonalize θ^1 and θ^2: namely, we use the well-known result that this is the case, provided one of the two bilinear forms – say, θ^1 – is invertible and the product $(\theta^1)^{-1}\theta^2$ is diagonalizable (see for example [R. Horn, C. Johnson, Matrix Analysis, pp. 228–234]).

So, if θ^1 and θ^2 are not proportional to each other and θ^1 is positive definite, then an appropriate choice of S will bring $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ into diagonal form with diagonal entries $(1, 1)$. Whence, the identity matrix. On the other hand, regardless of what form $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ might have achieved with this choice of S, it is still a symmetric matrix and hence diagonalizable. Actually, by means of a rotation

$$S = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix},$$

which is an element of the isotropy group at $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1} = \mathrm{diag}(1, 1)$, we can bring $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ into diagonal form which, under the assumption that θ^1 and θ^2 were not proportional at the outset, have different diagonal entries. Therefore, the theorem we have just quoted applies and we can see that the new diagonal entries of the matrices $\tilde{\theta}^i = T_{i1}(S^{-1})^{\mathrm{t}}\theta^1 S^{-1} + T_{i2}(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ $(i = 1, 2)$ can be chosen so that the product of T with the matrix M whose columns are the diagonal entries $(1, 1)$ of $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ and (a, d) of $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$, is equal to the identity matrix. Whence, the representative pair for this orbit is that listed under Type 4 in the statement. Besides, it is easy to see that the same argument applies if θ^1 was negative definite, since the isotropy group is still the same in this case.

The case that remains to be analyzed is that when θ^1 is nondegenerate, but nondefinite and θ^2 was not proportional to θ^1. With an appropriate $S \in \mathrm{GL}_2(\mathbb{R})$ we may assume that $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ is diagonal with diagonal entries $(1, -1)$. The isotropy group of this element is formed by the matrices of the Lorentz group and, by choosing

$$S = \begin{pmatrix} \cosh\omega & -\sinh\omega \\ -\sinh\omega & \cosh\omega \end{pmatrix},$$

it is easy to see that $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ will be diagonalizable by means of such a Lorentz transformation if and only if $\tanh 2\omega = \frac{2b}{a+c}$, where we originally had

$$(S^{-1})^{\mathrm{t}}\theta^2 S^{-1} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

This will obviously be the case if and only if the absolute value of $\frac{2b}{a+c}$ is strictly less than 1. But if this condition is fulfilled, then a T can be chosen as in the previous paragraph and therefore fall into Type 4.

Problems in the Lorentz-transformation argument arise when the absolute value of $\frac{2b}{a+c}$ is either strictly bigger than 1, or exactly equal to 1. In the first case we have a typical situation of two symmetric matrices that cannot be simultaneously diagonalized, but still have the chance of bringing the pair $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ and $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ into the representatives given in Type 6 of the statement. The condition that is definitely different, on the other hand, is that when $\frac{2b}{a+c}$ is equal to either $+1$ or to -1. In this case, $(S^{-1})^{\mathrm{t}}\theta^1 S^{-1}$ and $(S^{-1})^{\mathrm{t}}\theta^2 S^{-1}$ can only be brought into the representatives given in Type 7 of the statement.

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References

[1] C.P. Boyer, O.A. Sánchez-Valenzuela, Lie supergroups actions on supermanifolds, Trans. Am. Math. Soc. 323 (1991) 151–175.

[2] V. Guillemin, S. Sternberg, Supersymmetry and Equivariant de Rham Theory, Springer-Verlag, New York, 1999.

[3] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1977) 8–96.

[4] B. Kostant, Graded Manifolds, Graded Lie Theory and Prequantization, Lecture Notes in Mathematics, vol. 570, Springer-Verlag, 1977, pp. 177–306.

[5] Y.I. Manin, Gauge Field Theory and Complex Geometry, Springer-Verlag, New York, 1988.

[6] J. Monterde, O.A. Sánchez-Valenzuela, Existence and uniqueness of solutions to superdifferential equations, J. Geom. Phys. 10 (1993) 315–344.

[7] J. Monterde, J. Muñoz-Masqué, O.A. Sánchez-Valenzuela, Geometric properties of involutive distributions on graded manifolds, Indag. Math. 8 (1997) 217–246.

[8] G. Salgado, O.A. Sánchez-Valenzuela, Lie superalgebras based on $\mathfrak{gl}_n$ associated to the adjoint representation, and invariant geometric structures defined on them, Comm. Math. Phys. 241 (2003) 505–518.

[9] M. Scheunert, The Theory of Lie Superalgebras: An Introduction, Lecture Notes in Mathematics, vol. 716, Springer-Verlag, New York, 1979.

[10] O.A. Sánchez-Valenzuela, Linear supergroup actions. I. On the defining properties, Trans. Am. Math. Soc. 307 (1988) 569–595.

[11] V.S. Varadarajan, Supersymmetry for Mathematicians: An Introduction, American Mathematical Society, 2004.

[12] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. 17 (1982) 661–692.